Dynamic programming principle for one kind of stochastic recursive optimal control problem and Hamilton-Jacobi-Bellman equations*

Zhen Wu[a] zhiyong Yu[b]

School of Mathematics and System Sciences
Shandong University
Jinan 250100 P.R.China

Abstract. In this paper, we study one kind of stochastic recursive optimal control problem with the obstacle constraints for the cost function where the cost function is described by the solution of one reflected backward stochastic differential equations. We will give the dynamic programming principle for this kind of optimal control problem and show that the value function is the unique viscosity solution of the obstacle problem for the corresponding Hamilton-Jacobi-Bellman equations.

Keywords: Reflected backward stochastic differential equation, Recursive optimal control problem, Dynamic programming principle, Hamilton-Jacobi-Bellman equations, Viscosity solution.

AMS subject classification: 93E20, 60H10, 35K15

1. Introduction.

Nonlinear backward stochastic differential equations (BSDE in short) have been introduced by Pardoux & Peng [11]. Independently, Duffie & Epstein [6] introduced BSDE from economic background. In [6] they presented a stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate c_t but also on the future utility. Actually it corresponds to the solution of a particular BSDE associated with a generator which does not depend on the variable z. In mathematics the result in [11] is more general. Then El.Karoui, Peng and Quenez [10] gave some important properties of BSDE such as comparison theorem and applications in mathematical finance and optimal control theory. And also in this paper they gave the formulation of recursive utilities and their properties from the BSDE point of view. The recursive optimal control problem is presented as a kind of optimal control problem whose cost function is described by the solution of BSDE. In 1992, Peng [12] got the Bellman's dynamic programming principle for this kind of problem and proved the value function is a viscosity solution of one kind of quasi-linear second-order partial differential equation (PDE in short) which

* This work is supported by the National Natural Science Foundation (10671112) and the New century Young Teachers Program of Education Ministry , P.R.China.

[a] Email: wuzhen@sdu.edu.cn
[b] Email: yuzhiyong@sdu.edu.cn

is the well-known Hamilton-Jacobi-Bellman equation. Then in 1997, he virtually generalized these results to much more general situation, even under Non-Markvian framework. (See in [13]). In this chinese version, Peng used the backward semigroup property of BSDE to give a complete proof of the Bellman's dynamic programming principle for the recursive optimal problem introduced by a BSDE whose coefficient just satisfies Lipschitz condition, under Markovian and Non-Markovian framework. He also proved that the value function is a viscosity solution of a generalized Hamilton-Jacobi-Bellman equation.

Then El.Karoui, Kapoudjian, Pardoux, Peng and Quenez [9] studied the reflected BSDE with one barrier. The solution of the reflected BSDE is forced to stay above one given continuous stochastic process which is called "obstacle". For this purpose they introduced one increasing process to push the solution upwards in a kind of minimal way. They got the existence and uniqueness of the solution for this kind of reflected BSDE and also studied its relation with the obstacle problem for nonlinear parabolic PDE's within the Markov framework. Using two different methods, Snell envelope theory connected with fixed point principle and penalization method. Cvitanic and Karatzas [5] extended the result to reflected BSDE's with two barriers called upper and lower barriers, which are two given continuous processes. Hamadène and Lepeltier [7] generalized the results of El.Karoui et al [9] to one barrier which is right continuous and left upper semi-continuous. They used this model to solve the mixed optimal stochastic control problem when the terminal reward is only right continuous and left upper semi-continuous. In this kind of mixed control problem, the controller has two actions, one is of control and the other is of stopping his control strategy in view to maximize his payoff. Also in this paper Hamadène and Lepeltier generalized the result of Cvitanic and Karatzas ([5]) to reflected BSDE's with two barriers to processes S (lower barrier) and $-U$ (U is upper barrier) merely right continuous and left upper semicontinuous. And then Hamadène, Lepeltier and Wu [8] proved existence and uniqueness results of the solution for infinite horizon reflected backward stochastic differential equations with one or two barriers. They also apply those results to get the existence of optimal control strategy for the mixed control problem and a saddle-point strategy for the mixed game problem when, in both situation, the horizon is infinite.

In our paper, we study one kind of recursive optimal control problem with the obstacle constraints for the cost function. This means that the cost function of the control system is described by the solution of one reflected BSDE which is required to satisfy the obstacle constraints. This kind of the recursive optimal control problem has some practical sense such as, in financial market, the investor requires his recursive utility function value to be bigger than one specific function of his wealth. For this purpose, one increasing process is introduced to push the cost function value upward and we also hope this push power to be minimum. From the result in [9] and [7], we know that, in fact, this kind of problem is one mixed recursive optimal stochastic control problem.

One of our interesting problem is that if the dynamic programming principle still holds for the above optimal control problem. Using some properties of the reflected BSDE and analysis technique we give the positive answer for this question. This result can be seen as the generalized extension of the dynamic programming principle of the recursive control problem in [12] and [13] to the obstacle constraints case for the cost function. And then, we show that, provided the problem is formulated within a Markovian framework, the value function is the unique viscosity solution of the obstacle problem for one nonlinear parabolic PDEs which is called Hamilton-Jacobi-Bellman (HJB in short) equations.

The paper is organized as follows. In section 2, we present some preliminary results about re-

flected stochastic differential equations which play important role to study the dynamic programming principle of the optimal control problem. In section 3, we formulate the recursive optimal control problem with the obstacle constraints for the cost function and prove that the dynamic programming principle still holds. In section 4, we show that the value function of the control problem is the unique viscosity solution of the obstacle problem for corresponding HJB equations. In Appendix we put in some technique proof of the preliminary results of the reflected BSDE.

2. Preliminary results of the reflected BSDE

In this section, we give some preliminary results of the reflected BSDE which is useful to get the dynamic programming principle for the recursive optimal control problem with the obstacle constraints for the cost function.

Let $\{W_t, 0 \le t \le T\}$ be a $d-$dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t, 0 \le t \le T\}$ be the natural filtration of $\{W_t\}$, where $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and let $\mathcal{P}$ be the $\sigma-$algebra of predictable subsets of $\Omega \times [0,T]$.

Let us introduce some notation.

$$
\begin{aligned}
L^2 &= \left\{\xi \text{ is an } \mathcal{F}_T- \text{ measurable random variable s.t. } \mathbb{E}(|\xi|^2) < +\infty\right\}, \\
H^2 &= \left\{\{\varphi_t, 0 \le t \le T\} \text{ is a predictable process s.t. } \mathbb{E}\int_0^T |\varphi_t|^2 dt < +\infty\right\}, \\
S^2 &= \left\{\{\varphi_t, 0 \le t \le T\} \text{ is a predictable process s.t. } \mathbb{E}(\sup_{0\le t\le T} |\varphi_t|^2) < +\infty\right\}
\end{aligned}
$$

and the following reflected BSDE with one barrier:

$$
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \qquad 0 \le t \le T. \tag{2.1}
$$

Here $\xi \in L^2$, g is a map from $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ onto $\mathbb{R}$ satisfying

(i) $\forall\ (y,z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z) \in H^2$,

(ii) for some $L > 0$ and all $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, a.s.

$$
|g(t,y,z) - g(t,y',z')| \le L(|y-y'| - |z-z'|),
$$

an "obstacle" $\{S_t, 0 \le t \le T\}$, which is a continuous progressively measurable real-valued process satisfying

(iii) $\mathbb{E}\left(\sup_{0\le t\le T} |S_t|^2\right) < +\infty$.

Then from Theorem 5.2 in [9], there exists unique solution $\{(Y_t, Z_t, K_t), 0 \le t \le T\}$ taking values in $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}_+$, respectively, and satisfying:

(iv) $Y \in S^2$, $Z \in H^2$ and $K_T \in L^2$;

(v) $Y_t \ge S_t, \qquad 0 \le t \le T$;

(vi) $\{K_t\}$ is continuous and increasing, $K_0 = 0$ and

$$
\int_0^T (Y_t - S_t) dK_t = 0.
$$

Now we give two more accurate estimates on the norm of the solution similar to Proposition 3.5 and Proposition 3.6 in [9].

Proposition 2.1 *Let $\{(Y_t, Z_t, K_t), 0 \leq t \leq T\}$ be the solution of the above reflected BSDE, then there exists a constant C such that*

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq s\leq T} Y_s^2 + \int_t^T |Z_s|^2 + |K_T - K_t|^2\right\} \leq C\mathbb{E}^{\mathcal{F}_t}\left\{\xi^2 + \left(\int_t^T g(s,0,0)ds\right)^2 + \sup_{t\leq s\leq T} S_t^2\right\}.$$

This proposition is similar to Proposition 3.5 in [9]. However, the estimate is more precise which is necessary to get the desired results in next section. The proof is a little complicated and technical, some technique derive from [2], we put it in the Appendix.

And then, we need to estimate the variation of the solution induced by a variation of the reflected BSDE coefficients.

Proposition 2.2 *Let (ξ, g, S) and (ξ', g', S') be two triplets satisfying the above assumptions. Suppose (Y, Z, K) is the solution of the reflected BSDE (ξ, g, S) and (Y', Z', K') is the solution of the reflected BSDE (ξ', g', S'). Define*

$$\Delta\xi = \xi - \xi', \qquad \Delta g = g - g', \qquad \Delta S = S - S';$$

$$\Delta Y = Y - Y', \qquad \Delta Z = Z - Z', \qquad \Delta K = K - K'.$$

Then there exists a constant C such that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq s\leq T} |\Delta Y_s|^2 + \int_t^T |\Delta Z_s|^2 ds + |\Delta K_T - \Delta K_t|^2\right\} \\
&\leq C\mathbb{E}^{\mathcal{F}_t}\left\{|\Delta\xi|^2 + \left(\int_t^T |\Delta g(s, Y_s, Z_s)|ds\right)^2\right\} + C\left(\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq s\leq T} |\Delta S_s|^2\right\}\right)^{1/2} \Psi_{t,T}^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{t,T} = \mathbb{E}^{\mathcal{F}_t}\Bigg\{&|\xi|^2 + \left(\int_t^T |g(s,0,0)|ds\right)^2 + \sup_{t\leq s\leq T} |S_s|^2 \\
&+|\xi'|^2 + \left(\int_t^T |g'(s,0,0)|ds\right)^2 + \sup_{t\leq s\leq T} |S'_s|^2\Bigg\}.
\end{aligned}$$

The estimate of this proposition is more accurate than that in Proposition 3.6 in [9]. We also put the proof in the Appendix.

3. Formulation of the problem and Dynamic programming principle

In this section, we first formulate one kind of stochastic recursive optimal control problem with the obstacle constraints for the cost function, and then we prove that dynamic programming principle still holds for this kind of optimization problem.

We introduce the admissible control set $\mathcal{U}$ defined by

$$\mathcal{U} := \left\{ v(\cdot) \in H^2 | \quad v(\cdot) \quad \text{take value in } U \subset \mathbb{R}^k \right\}.$$

U is a compact set, the element of $\mathcal{U}$ is called admissible control.

For given admissible control, we consider the following control system

$$\begin{cases} dX_s^{t,\zeta;v} = b(s, X_s^{t,\zeta;v}, v_s)ds + \sigma(s, X_s^{t,\zeta;v}, v_s)dW_s, \qquad s \in [t,T], \\ X_t^{t,\zeta;v} = \zeta, \end{cases} \tag{3.1}$$

here $t \geq 0$ is regarded as the initial time, $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ as the initial state, the mappings

$$b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \qquad \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$$

satisfy the following conditions:

(H3.1) b and σ are continuous in t;

(H3.2) For some $L > 0$, and all $x, x' \in \mathbb{R}^n$, $v, v' \in U$, a.s.

$$|b(t,x,v) - b(t,x',v')| + |\sigma(t,x,v) - \sigma(t,x',v')| \leq L(|x - x'| + |v - v'|).$$

Obviously, under above assumptions, for any $v(\cdot) \in \mathcal{U}$, control system (3.1) has a unique strong solution $\{X_s^{t,\zeta;v},\ 0 \leq t \leq s \leq T\}$, and we also have the following estimates:

Proposition 3.1 *For all $t \in [0,T]$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $v(\cdot), v'(\cdot) \in \mathcal{U}$,*

$$\mathbb{E}^{\mathcal{F}_t} \left\{ \sup_{t \leq s \leq T} |X_s^{t,\zeta;v}|^2 \right\} \leq C(1 + |\zeta|^2); \tag{3.2}$$

$$\mathbb{E}^{\mathcal{F}_t} \left\{ \sup_{t \leq s \leq T} |X_s^{t,\zeta;v} - X_s^{t,\zeta';v'}|^2 \right\} \leq C|\zeta - \zeta'|^2 + C\mathbb{E}^{\mathcal{F}_t} \left\{ \int_t^T |v_s - v'_s|^2 ds \right\}, \tag{3.3}$$

where the constant C depends only on L.

Proposition 3.2 *For all $t \in [0,T]$, $x \in \mathbb{R}^n$, $v(\cdot) \in \mathcal{U}$, $\delta \in [0, T-t]$,*

$$\mathbb{E} \left\{ \sup_{t \leq s \leq t+\delta} |X_s^{t,x;v} - x|^2 \right\} \leq C\delta, \tag{3.4}$$

where the constant C depend only on x and L.

Now for any given admissible control $v(\cdot) \in \mathcal{U}$, we consider the following reflected BSDE

$$\begin{aligned} Y_s^{t,\zeta;v} &= \Phi(X_T^{t,\zeta;v}) + \int_s^T g(r, X_r^{t,\zeta;v}, Y_r^{t,\zeta;v}, Z_r^{t,\zeta;v}, v_r)dr \\ &\quad + K_T^{t,\zeta;v} - K_s^{t,\zeta;v} - \int_s^T Z_r^{t,\zeta;v} dW_r, \qquad t \leq s \leq T, \end{aligned} \tag{3.5}$$

here

$$\Phi = \Phi(x) : \mathbb{R}^n \to \mathbb{R}, \quad h = h(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R},$$
$$g = g(t,x,y,z,v) : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}$$

satisfy the following conditions:

(H3.3) g and h are continuous in t;

(H3.4) For some $L > 0$, and all $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$ $v, v' \in U$, a.s.

$$\begin{aligned}&|g(t,x,y,z,v) - g(t,x',y',z',v')| + |\Phi(x) - \Phi(x')| + |h(t,x) - h(t,x')|\\ &\leq L(|x-x'| + |y-y'| + |z-z'| + |v-v'|).\end{aligned}$$

Then from Theorem 5.2 in [9], there exists a unique triple $(Y^{t,\zeta;v}, Z^{t,\zeta;v}, K^{t,\zeta;v})$, which is the solution of reflected BSDE (3.5), satisfying

(i) $Y^{t,\zeta;v} \in S^2$, $Z^{t,\zeta;v} \in H^2$ and $K_T^{t,\zeta;v} \in L^2$;

(ii) $Y_s^{t,\zeta;v} \geq h(s, X_s^{t,\zeta;v})$, $t \leq s \leq T$;

(iii) $\{K_s^{t,\zeta;v}\}$ is increasing and continuous, $K_t^{t,\zeta;v} = 0$, and $\int_t^T (Y_s^{t,\zeta;v} - h(s, X_s^{t,\zeta;v}))dK_s^{t,\zeta;v} = 0$.

Moreover, we can get the following estimates for the solution of (3.5) from Proposition 2.1 and 2.2.

Proposition 3.3

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq s\leq T}|Y_s^{t,\zeta;v}|^2 + \int_t^T |Z_s^{t,\zeta;v}|^2 ds + |K_T^{t,\zeta;v}|^2\right\} \leq C(1+|\zeta|^2). \tag{3.6}$$

Proposition 3.4

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq s\leq T}|Y_s^{t,\zeta;v} - Y^{t,\zeta';v'}|^2 + \int_t^T |Z_s^{t,\zeta;v} - Z_s^{t,\zeta';v'}|^2 ds + |K_T^{t,\zeta;v} - K_T^{t,\zeta';v'}|^2\right\}\\ &\leq C|\zeta-\zeta'|^2 + C\mathbb{E}^{\mathcal{F}_t}\left\{\int_t^T |v_s - v'_s|^2 ds\right\}\\ &\quad + C(1+|\zeta|+|\zeta'|)\left(|\zeta-\zeta'|^2 + \mathbb{E}^{\mathcal{F}_t}\left\{\int_t^T |v_s - v'_s|^2 ds\right\}\right)^{1/2}.\end{aligned} \tag{3.7}$$

Given the control process $v(\cdot) \in \mathcal{U}$, we introduce the associated cost functional:

$$J(t,x;v(\cdot)) := Y_s^{t,x;v}|_{s=t}, \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.8}$$

and we define the value function of the stochastic optimal control problem

$$u(t,x) := ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n. \tag{3.9}$$

This is one kind of stochastic recursive optimal control problem with the obstacle constraints for the cost function: $Y_s^{t,x;v} \geq h(s, X_s^{t,x;v})$, $t \leq s \leq T$. In financial market, if $X_s^{t,x;v}$ represents the wealth

of the one investor , $Y_s^{t,x;v}$: the recursive utility cost function, the constraint is that the investor requires his cost function value to be bigger than one function of his wealth at any time.

Remark 3.5 From Proposition 2.3 in [9] and the definition in [7] and [8], we know that the above optimal control problem is one recursive mixed optimal control problem:

$$u(t,x) := ess \sup_{v(\cdot)\in\mathcal{U}} Y_t^{t,x;v} = ess \sup_{v(\cdot)\in\mathcal{U}} ess \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{\mathcal{F}_t}\left\{\int_t^\tau g(s,X_s^{t,\zeta;v},Y_s^{t,\zeta;v},Z_s^{t,\zeta;v},v_s)ds \right.$$
$$\left. +h(X_\tau^{t,\zeta;v})1_{\tau<T} + \Phi(X_T^{t,\zeta;v})1_{\tau=T}\right\}$$

where $\mathcal{T}$ is the set of all stopping times dominated by T and $\mathcal{T}_t = \{\tau\in\mathcal{T};\quad t\le\tau\le T\}$.

In this kind of recursive mixed control problem, the controller has two actions, one is of control $v(\cdot)$ and the other is of stopping his control strategy in view to maximize his recursive payoff. The more detail about this kind of problem can be seen in [9], [7] and [8].

Now we continue to study the former control problem (3.9) and show that celebrated dynamic programming principle still holds for this kind of optimization problem. The main proof idea comes from the proof of dynamic programming principle for recursive problem given by Peng in chinese version [13].

For each $t > 0$, we denote by $\{\mathcal{F}_s^t, t\le s\le T\}$ the natural filtration of the Brownian motion $\{W_s - W_t, t\le s\le T\}$, augmented by the P-null sets of $\mathcal{F}$ and we introduce the following subspaces of admissible controls

$$\mathcal{U}^t := \{v(\cdot)\in\mathcal{U}\quad|\quad v(s) \text{ is } \{\mathcal{F}_s^t\} \text{ progressively measurable}, \forall\, t\le s\le T.\}$$
$$\bar{\mathcal{U}}^t := \left\{v_s = \sum_{j=1}^N v_s^j 1_{A_j}\quad|\quad v_s^j\in\mathcal{U}^t,\quad \{A_j\}_{j=1}^N \text{ is a partition of } (\Omega,\mathcal{F}_t).\right\}$$

Firstly we will show that

Proposition 3.6 *Under the assumptions (H3.1)–(H3.4), the value function $u(t,x)$ defined in (3.9) is a deterministic function.*

Proof: Firstly , we will show

$$ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) = ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)). \tag{3.10}$$

Obviously,

$$ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) \ge ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)).$$

we need to show the inverse inequality. $\forall\varepsilon>0$, there exists $\tilde{v}(\cdot)\in\mathcal{U}$ such that

$$P\left\{J(t,x;\tilde{v}(\cdot)) > ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) - \varepsilon\right\} = \delta > 0.$$

From (3.7), we know $\forall \bar{v}(\cdot) \in \bar{\mathcal{U}}^t$,

$$\mathbb{E}\left\{|Y_t^{t,x;\bar{v}} - Y_t^{t,x;\tilde{v}}|^2\right\} \leq C\mathbb{E}\int_t^T |\bar{v}_s - \tilde{v}_s|^2 ds + C(1+x)\left(\mathbb{E}\int_t^T |\bar{v}_s - \tilde{v}_s|^2 ds\right)^{1/2}.$$

Note that $\bar{\mathcal{U}}^t$ is dense in $\mathcal{U}$, then there exists a sequence $\{v_n(\cdot)\}_{n=1}^{\infty} \in \bar{\mathcal{U}}^t$ such that

$$\lim_{n\to\infty} \mathbb{E}\left\{|Y_t^{t,x;v_n} - Y_t^{t,x;\tilde{v}}|^2\right\} = 0.$$

Then, there exists a subsequence, we denote without loss of generality $\{v_n(\cdot)\}_{n=1}^{\infty}$ also, such that

$$\lim_{n\to\infty} Y_t^{t,x;v_n} = Y_t^{t,x;\tilde{v}} \quad a.s.,$$

then

$$\begin{aligned}
&P\left(\bigcap_{m=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{|Y_t^{t,x;v_n} - Y_t^{t,x;\tilde{v}}| < \frac{1}{m}\right\}\right) = 1,\\
&P\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left\{|Y_t^{t,x;v_n} - Y_t^{t,x;\tilde{v}}| < \frac{1}{m}\right\}\right) = 1, \qquad \forall m \in \mathbb{N},\\
&\lim_{N\to\infty} P\left(\bigcap_{n=N}^{\infty}\left\{|Y_t^{t,x;v_n} - Y_t^{t,x;\tilde{v}}| < \frac{1}{m}\right\}\right) = 1, \qquad \forall m \in \mathbb{N},\\
&\lim_{N\to\infty} P\left\{|Y_t^{t,x;v_n} - Y_t^{t,x;\tilde{v}}| < \frac{1}{m}\right\} = 1, \qquad \forall m \in \mathbb{N}.
\end{aligned}$$

We select m big enough such that $1/m < \varepsilon$ and denote

$$\begin{aligned}
A &= \left\{\omega | Y_t^{t,x;\tilde{v}} > ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) - \varepsilon\right\};\\
B_N &= \left\{\omega || Y_t^{t,x;v_N} - Y_t^{t,x;\tilde{v}}| \leq \frac{1}{m}\right\}, \quad N = 1, 2, \cdots,
\end{aligned}$$

then, from above definition, $P(A) = \delta > 0$ and $\lim_{N\to\infty} P(B_N) = 1$. We select N big enough such that $P(B_N) > 1 - \delta$, then

$$P(AB_N) = P(A) + P(B_N) - P(A \cup B_N) > \delta + (1-\delta) - 1 = 0.$$

It is easily to check

$$P\left\{Y_t^{t,x;v_N} > ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) - 2\varepsilon\right\} \geq P(AB_N) > 0.$$

This inequality implies

$$ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)) \geq ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)) - 2\varepsilon.$$

From the arbitrariness of ε, we get

$$ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)) \geq ess \sup_{v(\cdot)\in\mathcal{U}} J(t,x;v(\cdot)).$$

Then we obtain (3.10).

Secondly, we will show

$$ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)) = ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot)) \tag{3.11}$$

Obviously,

$$ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)) \geq ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot)).$$

We need to show the inverse inequality also.

Let us admit for a moment the following lemma. The main idea of the lemma is to consider the partition of probability space, which is first introduced by Theorem 4.7 in [13].

Lemma 3.7

$$X_\cdot^{t,x;\sum_{j=1}^N v^j 1_{A_j}} = \sum_{j=1}^N 1_{A_j} X_\cdot^{t,x;v^j}; \quad Y_\cdot^{t,x;\sum_{j=1}^N v^j 1_{A_j}} = \sum_{j=1}^N 1_{A_j} Y_\cdot^{t,x;v^j};$$
$$Z_\cdot^{t,x;\sum_{j=1}^N v^j 1_{A_j}} = \sum_{j=1}^N 1_{A_j} Z_\cdot^{t,x;v^j}; \quad K_\cdot^{t,x;\sum_{j=1}^N v^j 1_{A_j}} = \sum_{j=1}^N 1_{A_j} K_\cdot^{t,x;v^j}.$$

$\forall v(\cdot) \in \bar{\mathcal{U}}^t$, we have

$$J(t,x;v(\cdot)) = J(t,x;\sum_{j=1}^N v^j(\cdot)1_{A_j}) = \sum_{j=1}^N 1_{A_j} J(t,x;v^j(\cdot)).$$

Note that $v^j(\cdot)$ are $\{\mathcal{F}_s^t\}$ progressively measurable, then $J(t,x;v^j(\cdot))$ $(j=1,2,\cdots,N)$ are deterministic. Without loss of generality, we assume that

$$J(t,x;v^1(\cdot)) \geq J(t,x;v^j(\cdot)), \quad \forall j=2,3,\cdots,N.$$

So that

$$J(t,x;v(\cdot)) \leq J(t,x;v^1(\cdot)) \leq ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot)).$$

From the arbitrariness of $v(\cdot)$, we get

$$ess \sup_{v(\cdot)\in\bar{\mathcal{U}}^t} J(t,x;v(\cdot)) \leq ess \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot)),$$

and obtain (3.11).

However, when $v(\cdot) \in \mathcal{U}^t$, the cost functional $J(t,x;v(\cdot))$ is deterministic, so

$$u(t,x) = \sup_{v(\cdot)\in\mathcal{U}^t} J(t,x;v(\cdot))$$

is deterministic and the proof is completed. $\square$

We need to give

Proof of Lemma 3.7 For every $j = 1,2,\cdots,N$, we denote

$$(X_s^j, Y_s^j, Z_s^j, K_s^j) \equiv (X_s^{t,x;v^j}, Y_s^{t,x;v^j}, Z_s^{t,x;v^j}, K_s^{t,x;v^j}).$$

X^j is the solution of the following stochastic differential equations:

$$X_s^j = x + \int_t^s b(r, X_r^j, v_r^j)dr + \int_t^s \sigma(r, X_r^j, v_r^j), \quad s \in [t,T].$$

(Y^j, Z^j, K^j) satisfies the following reflected BSDE:

$$\begin{aligned} &Y_s^j = \Phi(X_T^j) + \int_s^T g(r, X_r^j, Y_r^j, Z_r^j, v_r)dr + K_T^j - K_s^j - \int_s^T Z_r^j dW_r, \quad s \in [t,T]; \\ &Y_s^j \geq h(s, X_s^j), \quad s \in [t,T]; \quad \int_t^T (Y_s^j - h(s, X_s^j))dK_s^j = 0. \end{aligned}$$

We multiply 1_{A_j} on the both sides of the above equations, then sum the equations. From the trivial fact:

$$\sum_j 1_{A_j}\varphi(x_j) = \varphi(\sum_j x_j 1_{A_j}),$$

we get

$$\begin{aligned} \sum_{j=1}^N 1_{A_j} X_s^j &= x + \int_t^s b(r, \sum_{j=1}^N 1_{A_j} X_r^j, \sum_{j=1}^N 1_{A_j} v_r^j)dr + \int_t^s \sigma(r, \sum_{j=1}^N 1_{A_j} X_r^j, \sum_{j=1}^N 1_{A_j} v_r^j)dW_r; \\ \sum_{j=1}^N 1_{A_j} Y_s^j &= \Phi(\sum_{j=1}^N 1_{A_j} X_T^j) + \int_s^T g(r, \sum_{j=1}^N 1_{A_j} X_r^j, \sum_{j=1}^N 1_{A_j} Y_r^j, \sum_{j=1}^N 1_{A_j} Z_r^j, \sum_{j=1}^N 1_{A_j} v_r^j)dr \\ &\quad + \sum_{j=1}^N 1_{A_j} K_T^j - \sum_{j=1}^N 1_{A_j} K_s^j - \int_s^T \sum_{j=1}^N 1_{A_j} Z_r^j; \\ \sum_{j=1}^N 1_{A_j} Y_s^j &\geq h(s, \sum_{j=1}^N 1_{A_j} X_s^j); \quad \int_t^T \left(\sum_{j=1}^N 1_{A_j} Y_s^j - h(s, \sum_{j=1}^N 1_{A_j} X_s^j) \right) d\left(\sum_{j=1}^N 1_{A_j} K_s^j \right) = 0. \end{aligned}$$

Then from the uniqueness of the solution of stochastic differential equations and reflected BSDE, we get the conclusion. $\square$

We next will discuss the continuity of value function $u(t,x)$ with respect to x. We have the following estimation:

Lemma 3.8 *For each $t \in [0,T]$, x and $x' \in \mathbb{R}^n$, we have*
(i) $|u(t,x) - u(t,x')|^2 \le C|x-x'|^2 + C(1+|x|+|x'|)|x-x'|$;
(ii)$|u(t,x)| \le C(1+|x|)$.

Proof: From estimation (3.6) and (3.7), for each admissible control $v(\cdot) \in \mathcal{U}$, we have

$$|J(t,x;v(\cdot))| \le C(1+|x|);$$

$$|J(t,x;v(\cdot)) - J(t,x';v(\cdot))|^2 \le C|x-x'|^2 + C(1+|x|+|x'|)|x-x'|. \tag{3.12}$$

On the other hand, for each $\varepsilon > 0$, there exist $v(\cdot)$ and $v'(\cdot) \in \mathcal{U}$ such that

$$\begin{aligned} J(t,x;v'(\cdot)) &\le u(t,x) \le J(t,x;v(\cdot)) + \varepsilon, \\ J(t,x';v(\cdot)) &\le u(t,x') \le J(t,x';v'(\cdot)) + \varepsilon. \end{aligned}$$

Then from the estimation of J, we get

$$-C(1+|x|) \le J(t,x;v'(\cdot)) \le u(t,x) \le J(t,x;v(\cdot)) + \varepsilon \le C(1+|x|) + \varepsilon.$$

From the arbitrariness of ε, we obtain (ii). Similarly,

$$J(t,x;v'(\cdot)) - J(t,x';v'(\cdot)) - \varepsilon \le u(t,x) - u(t,x') \le J(t,x;v(\cdot)) - J(t,x';v(\cdot)) + \varepsilon,$$

$$\begin{aligned} &|u(t,x) - u(t,x')| \\ &\le \max\{|J(t,x;v(\cdot)) - J(t,x';v(\cdot))|, |J(t,x;v'(\cdot)) - J(t,x';v'(\cdot))|\} + \varepsilon, \end{aligned}$$

$$\begin{aligned} &|u(t,x) - u(t,x')|^2 \\ &\le 2\max\left\{|J(t,x;v(\cdot)) - J(t,x';v(\cdot))|^2, |J(t,x;v'(\cdot)) - J(t,x';v'(\cdot))|^2\right\} + 2\varepsilon^2 \\ &\le 2C|x-x'|^2 + 2C(1+|x|+|x'|)|x-x'| + 2\varepsilon^2. \end{aligned}$$

The we can obtain (i). □

We also have

Lemma 3.9 *$\forall t \in [0,T]$, $\forall v(\cdot) \in \mathcal{U}$, for all $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have*

$$J(t,\zeta;v(\cdot)) = Y_t^{t,\zeta;v}.$$

Proof: We first study the simple case: ζ has the following form:

$$\zeta = \sum_{i=1}^{N} 1_{A_i} x_i,$$

where $\{A\}_{i=1}^N$ is a finite partition of $(\Omega, \mathcal{F}_t)$, and $x_i \in \mathbb{R}^n$, for $1 \le i \le N$. The similar argument as Lemma 3.7 leads to

$$Y_s^{t,\zeta;v} = Y_s^{t,\sum_{i=1}^N 1_{A_i} x_i;v} = \sum_{i=1}^N 1_{A_i} Y_s^{t,x_i;v}, \qquad s \in [t,T].$$

From the definition (3.8), we deduce that

$$Y_t^{t,\zeta;v} = \sum_{i=1}^N 1_{A_i} Y_t^{t,x_i;v} = \sum_{i=1}^N 1_{A_i} J(t, x_i; v(\cdot)) = J(t, \sum_{i=1}^N 1_{A_i} x_i; v(\cdot)) = J(t, \zeta; v(\cdot)).$$

Therefore, for simple functions, we have the desired result.

Given a general $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we can choose a sequence of simple functions $\{\zeta_i\}$ which converges to ζ in $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, from the estimate (3.7) and (3.12), we have

$$\begin{aligned}
&\mathbb{E}\left\{|Y_t^{t,\zeta;v} - Y_t^{t,\zeta_i;v}|^2\right\} \\
&\le \mathbb{E}\left\{C|\zeta - \zeta_i|^2 + C(1 + |\zeta| + |\zeta_i|)|\zeta - \zeta_i|\right\} \\
&\le C\mathbb{E}\left\{|\zeta - \zeta_i|^2\right\} + C\left(\mathbb{E}\left\{(1 + |\zeta| + |\zeta_i|)^2\right\}\right)^{1/2}\left(\mathbb{E}\left\{|\zeta - \zeta_i|^2\right\}\right)^{1/2} \\
&\to 0, \qquad as\ i \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left\{|J(t, \zeta; v(\cdot)) - J(t, \zeta_i; v(\cdot))|^2\right\} \\
&\le \mathbb{E}\left\{C|\zeta - \zeta_i|^2 + C(1 + |\zeta| + |\zeta_i|)|\zeta - \zeta_i|\right\} \\
&\le C\mathbb{E}\left\{|\zeta - \zeta_i|^2\right\} + C\left(\mathbb{E}\left\{(1 + |\zeta| + |\zeta_i|)^2\right\}\right)^{1/2}\left(\mathbb{E}\left\{|\zeta - \zeta_i|^2\right\}\right)^{1/2} \\
&\to 0, \qquad as\ i \to \infty,
\end{aligned}$$

and $Y_t^{t,\zeta_i;v} = J(t, \zeta_i; v(\cdot))$, the proof is completed. □

For the value function of our recursive optimal control problem, we have

Lemma 3.10 *Fixed $t \in [0,T)$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, for each $v(\cdot) \in \mathcal{U}$, we have*

$$u(t, \zeta) \ge Y_t^{t,\zeta;v}. \tag{3.13}$$

On the other hand, for each $\varepsilon > 0$, there exists an admissible control $v(\cdot) \in \mathcal{U}$ such that

$$u(t, \zeta) \le Y_t^{t,\zeta;v} + \varepsilon, \quad a.s.. \tag{3.14}$$

Proof: We first prove (3.13). When ζ is a simple function:

$$\zeta = \sum_{i=1}^N 1_{A_i} x_i,$$

for all $v(\cdot) \in \mathcal{U}$, we have

$$Y_t^{t,\zeta;v} = Y_t^{t,\sum_{i=1}^N 1_{A_i} x_i;v} = \sum_{i=1}^N 1_{A_i} Y_t^{t,x_i;v} \le \sum_{i=1}^N 1_{A_i} u(t,x_i) = u(t,\zeta).$$

When $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we can choose a sequence of simple functions $\{\zeta_i\}$ which converges to ζ in $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, similarly with Lemma 3.9, we have

$$\mathbb{E}\left\{|Y_t^{t,\zeta;v} - Y_t^{t,\zeta_i;v}|^2\right\} \to 0; \quad \mathbb{E}\left\{|u(t,\zeta) - u(t,\zeta_i)|^2\right\} \to 0.$$

Then, there exists a subsequence, we use same notation without loss of generality also, such that

$$\lim_{i\to\infty} Y_t^{t,\zeta_i;v} = Y_t^{t,\zeta;v}, \qquad a.s. \quad \lim_{i\to\infty} u(t,\zeta_i) = u(t,\zeta), \qquad a.s.$$

here $Y_t^{t,\zeta_i;v} \le u(t,\zeta_i)$, $i = 1, 2, \cdots$, so $Y_t^{t,\zeta;v} \le u(t,\zeta)$.

We turn to prove (3.14). We first deal with the case that ζ is a bounded random variable: $\zeta \in L^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. We suppose that $|\zeta| \le M$ and construct a simple random variable $\eta \in L^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$

$$\eta = \sum_{i=1}^N 1_{A_i} x_i$$

such that

(i) $|\eta| \le |\zeta|$;

(ii) $|\eta - \zeta| \le \min\left\{\frac{\varepsilon}{6\sqrt{C}}, \frac{\varepsilon^2}{36C(1+2M)}\right\}$.

For any $v(\cdot) \in \mathcal{U}$, we have

$$|Y_t^{t,\zeta;v} - Y_t^{t,\eta;v}| \le \frac{\varepsilon}{3}; \quad |u(t,\zeta) - u(t,\eta)| \le \frac{\varepsilon}{3}.$$

Then for each x_i, we can choose an $\{\mathcal{F}_s^t\}$−adapted admissible control $v^i(\cdot)$ such that

$$u(t,x_i) \le Y^{t,x_i;v_i} + \frac{\varepsilon}{3}.$$

We denote

$$v(\cdot) := \sum_{i=1}^N 1_{A_i} v^i(\cdot),$$

then

$$\begin{aligned} Y_t^{t,\zeta;v} &\ge -|Y_t^{t,\zeta;v} - Y_t^{t,\eta;v}| + Y_t^{t,\eta;v} \ge -\frac{\varepsilon}{3} + \sum_{i=1}^N 1_{A_i} Y_t^{t,x_i;v_i} \\ &\ge -\frac{\varepsilon}{3} + \sum_{i=1}^N 1_{A_i}(u(t,x_i) - \frac{\varepsilon}{3}) = -\frac{2}{3}\varepsilon + u(t,\eta) \\ &\ge -\varepsilon + u(t,\zeta). \end{aligned}$$

Therefore, for $\zeta \in L^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the desired result (3.14).

Given a general $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we note that ζ have the following form:

$$\zeta = \sum_{i=1}^{\infty} 1_{A_i} \zeta_i,$$

where $\{A_i\}_{i=1}^{\infty}$ is a partition of $(\Omega, \mathcal{F}_t)$, $x_i \in \mathbb{R}^n$ $(i = 1, 2, \cdots)$, $|\zeta_i| \le i$ and $\zeta_i \in L^\infty(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. So, for every ζ_i, there exists $v^i(\cdot) \in \mathcal{U}$, such that

$$u(t, \zeta_i) \le Y^{t,\zeta_i;v_i} + \varepsilon.$$

We denote

$$v(\cdot) = \sum_{i=1}^{\infty} 1_{A_i} v^i(\cdot),$$

and then

$$\begin{aligned} u(t,\zeta) = u(t, \sum_{i=1}^{\infty} 1_{A_i}\zeta_i) &= \sum_{i=1}^{\infty} 1_{A_i} u(t,\zeta_i) \le \sum_{i=1}^{\infty} 1_{A_i}(Y^{t,\zeta_i;v_i} + \varepsilon) \\ &= \sum_{i=1}^{\infty} 1_{A_i} Y^{t,\zeta_i;v_i} + \varepsilon = Y_t^{t,\zeta;v} + \varepsilon. \end{aligned}$$

The proof is completed. $\square$

Now we start to discuss the (generalized) dynamic programming principle for our recursive optimal control problem (3.9). In [13], Peng first used the idea of (backward) semigroups of BSDE to prove the dynamic programming principle for the recursive optimal control problem associated to BSDE.

Firstly we introduce a family of (backward) semigroups which come from Peng's idea [13].

Given the initial condition (t, x), an admissible control $v(\cdot) \in \mathcal{U}$, a positive number $\delta \le T - t$ and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we denote

$$G_{t,t+\delta}^{t,x;v}[\eta] := Y_t,$$

where $(Y_s, Z_s, K_s)_{t \le s \le t+\delta}$ is the solution of the following reflected BSDE with time horizon $t + \delta$

$$\begin{aligned} Y_s = \eta + &\int_s^{t+\delta} g(r, X_r^{t,x;v}, Y_r, Z_r, v_r)dr + K_{t+\delta} - K_s \\ &- \int_s^{t+\delta} Z_r dW_r, \qquad t \le s \le t + \delta, \end{aligned}$$

satisfying

(i) $Y \in S^2$, $Z \in H^2$ and $K_{t+\delta} \in L^2$;

(ii) $Y_s \ge h(s, X_s^{t,x;v})$, $t \le s \le t + \delta$;

(iii) $\{K_s\}$ is increasing and continuous, $K_t = 0$, $\int_t^{t+\delta} (Y_s - h(s, X_s^{t,x;v}))dK_s = 0$.

Obviously,

$$G_{t,T}^{t,x;v}[\Phi(X_T^{t,x;v})] = G_{t,t+\delta}^{t,x;v}[Y_{t+\delta}^{t,x;v}].$$

Then our (generalized) dynamic programming principle holds.

Theorem 3.11 *Under the assumptions (H3.1)–(H3.4), the value function $u(t,x)$ obeys the following dynamic programming principle: For each $0<\delta\leq T-t$,*

$$u(t,x)=ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[u(t+\delta,X^{t,x;v}_{t+\delta})]. \tag{3.15}$$

Proof: We have

$$\begin{aligned}u(t,x)&=ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,T}[\Phi(X^{t,x;v}_T)]=ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[Y^{t,x;v}_{t+\delta}]\\&=ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[Y^{t+\delta,X^{t,x;v}_{t+\delta};v}_{t+\delta}].\end{aligned}$$

From Lemma 3.10 and the comparison theorem of reflected BSDE (Theorem 4.1 in [9]),

$$u(t,x)\leq ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[u(t+\delta,X^{t,x;v}_{t+\delta})].$$

On the other hand, for every $\varepsilon>0$, we can find an admissible control $\bar{v}(\cdot)\in\mathcal{U}$ such that

$$u(t+\delta,X^{t,x;v}_{t+\delta})\leq Y^{t+\delta,X^{t,x;v}_{t+\delta};\bar{v}}_{t+\delta}+\varepsilon.$$

From this and the comparison theorem, we get

$$u(t,x)\geq ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[u(t+\delta,X^{t,x;v}_{t+\delta})-\varepsilon].$$

From Proposition 2.2, there exists a positive constant C_0 such that

$$u(t,x)\geq ess\sup_{v(\cdot)\in\mathcal{U}}G^{t,x;v}_{t,t+\delta}[u(t+\delta,X^{t,x;v}_{t+\delta})]-C_0\varepsilon.$$

Therefore, letting $\varepsilon\downarrow 0$, we obtain the equation (3.15). $\square$

At the end of this section, we devote ourselves to obtaining the continuity of $u(t,x)$ with respect to t.

Proposition 3.12 *The value function $u(t,x)$ is continuous in t.*

Proof: We define $Y^{t,x;v}_s$ for all $s\in[0,T]$ by choosing $Y^{t,x;v}_s\equiv Y^{t,x;v}_t$ for $0\leq s\leq t$. And we define the "obstacle"

$$S^{t,x;v}_s=\begin{cases}h(s,X^{t,x;v}_s); & t\leq s\leq T;\\ h(t,x); & 0\leq s\leq t.\end{cases}$$

Fixed $x\in\mathbb{R}^n$, for all $0\leq t_1\leq t_2\leq T$, we analysis the difference of $u(t_1,x)$ and $u(t_2,x)$.
$\forall\varepsilon>0$, there exist $v_1(\cdot)\in\mathcal{U}$, $v_2(\cdot)\in\mathcal{U}$, such that

$$Y^{t_1,x;v_2}_{t_1}\leq u(t_1,x)\leq Y^{t_1,x;v_1}_{t_1}+\varepsilon;\quad Y^{t_2,x;v_1}_{t_2}\leq u(t_2,x)\leq Y^{t_2,x;v_2}_{t_2}+\varepsilon.$$

Then,

$$Y_{t_1}^{t_1,x;v_2} - Y_{t_2}^{t_2,x;v_2} - \varepsilon \le u(t_1,x) - u(t_2,x) \le Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1} + \varepsilon,$$

$$|u(t_1,x) - u(t_2,x)| \le \max\{|Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1}|, |Y_{t_1}^{t_1,x;v_2} - Y_{t_2}^{t_2,x;v_2}|\} + \varepsilon.$$

Here we only estimate $|Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1}|$ and the estimate of $|Y_{t_1}^{t_1,x;v_2} - Y_{t_2}^{t_2,x;v_2}|$ is same. From Proposition 2.2, we have

$$\begin{aligned}
&|Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1}|^2 = |Y_0^{t_1,x;v_1} - Y_0^{t_2,x;v_1}|^2 \\
&\le \mathbb{E}\left\{\sup_{0\le s\le T} |Y_s^{t_1,x;v_1} - Y_s^{t_2,x;v_1}|^2\right\} \\
&\le C\mathbb{E}\left\{|\Phi(X_T^{t_1,x;v_1}) - \Phi(X_T^{t_2,x;v_1})|^2\right\} \\
&\quad + C\mathbb{E}\left\{\left(\int_0^T |1_{[t_1,T]} g(s, X_s^{t_1,x;v_1}, Y_s^{t_1,x;v_1}, Z_s^{t_1,x;v_1}, v_1(s))\right.\right. \\
&\qquad\qquad \left.\left. - 1_{[t_2,T]} g(s, X_s^{t_2,x;v_1}, Y_s^{t_1,x;v_1}, Z_s^{t_1,x;v_1}, v_1(s))|ds\right)^2\right\} \\
&\quad + C\Psi_{0,T}^{1/2}\left(\mathbb{E}\left\{\sup_{0\le s\le T} |S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2\right\}\right)^{1/2},
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
\Psi_{0,T} = \mathbb{E}\Bigg\{ &|\Phi(X_T^{t_1,x;v_1})|^2 + \left(\int_{t_1}^T |g(s, X_s^{t_1,x;v_1}, 0, 0, v_1(s))|ds\right)^2 \\
&+ \sup_{t_1\le s\le T} |h(s, X_s^{t_1,x;v_1})|^2 + |\Phi(X_T^{t_2,x;v_1})|^2 \\
&+ \left(\int_{t_2}^T |g(s, X_s^{t_2,x;v_1}, 0, 0, v_1(s))|ds\right)^2 + \sup_{t_2\le s\le T} |h(s, X_s^{t_2,x;v_1})|^2 \Bigg\}.
\end{aligned}$$

Now we deal with the items for the right side of inequality (3.16).

The first item: From Lipschitz condition, Proposition 3.1 and Proposition 3.2, we get

$$I \le C\mathbb{E}\left\{|X_T^{t_1,x;v_1} - X_T^{t_2,x;v_1}|^2\right\} \le C\mathbb{E}\{|X_{t_1}^{t_2,x;v_1} - x|^2\} \le C(t_2 - t_1).$$

The second item: From Lipschitz condition, $(a+b)^2 \le a^2/2 + b^2/2$, Proposition 3.1, Proposition 3.2 and Proposition 3.3, we get

$$II \le C(t_2 - t_1).$$

The third item: As the same argument we get

$$\Psi_{0,T} \le C.$$

We next discuss

$$\begin{aligned}
|S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2 &= |h(s, X_s^{t_1,x;v_1}) - h(s, X_s^{t_2,x;v_1})|^2 \\
&\le C|X_s^{t_1,x;v_1} - X_s^{t_2,x;v_1}|^2; \qquad s \in [t_2, T],
\end{aligned}$$

$$
\begin{aligned}
|S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2 &= |h(s, X_s^{t_1,x;v_1}) - h(t_2,x)|^2 \\
&\leq C|X_s^{t_1,x;v_1} - x|^2 + 2|h(s,x) - h(t_2,x)|^2; \qquad s \in [t_1,t_2],
\end{aligned}
$$

$$
|S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2 = |h(t_1,x) - h(t_2,x)|^2; \qquad s \in [0,t_1].
$$

So we have

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{0\leq s\leq T} |S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2\right\} \\
&\leq \mathbb{E}\left\{\left(\sup_{0\leq s\leq t_1} + \sup_{t_1\leq s\leq t_2} + \sup_{t_2\leq s\leq T}\right)|S_s^{t_1,x;v_1} - S_s^{t_2,x;v_1}|^2\right\} \\
&\leq C(t_2-t_1) + |h(t_1,x) - h(t_2,x)|^2 + 2\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)|^2 \\
&\leq C(t_2-t_1) + 3\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)|^2.
\end{aligned}
$$

From the above analysis , we know

$$
|Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1}|^2 \leq C(t_2-t_1) + 3\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)|^2,
$$

$$
|Y_{t_1}^{t_1,x;v_1} - Y_{t_2}^{t_2,x;v_1}| \leq C(t_2-t_1)^{1/2} + 3\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)|.
$$

The same argument used to $|Y_{t_1}^{t_1,x;v_2} - Y_{t_2}^{t_2,x;v_2}|^2$ leads to

$$
|u(t_1,x) - u(t_2,x)| \leq C(t_2-t_1)^{1/2} + 3\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)| + \varepsilon.
$$

Because of the arbitrariness of ε, we get

$$
|u(t_1,x) - u(t_2,x)| \leq C(t_2-t_1)^{1/2} + 3\sup_{t_1\leq s\leq t_2}|h(s,x) - h(t_2,x)|.
$$

From the continuity of $h(t,x)$ with respect to t, we get the continuity of $u(t,x)$ with respect to t. The proof is completed. $\square$

4. Viscosity solution of an obstacle problem for HJB equations

In this section, we relate the value function of above recursive optimal control problem with the following obstacle problem for nonlinear second-order parabolic PDEs which is called Hamilton-Jacobi-Bellman equations:

$$
\begin{cases}
\min\big(u(t,x) - h(t,x), \\
\qquad -\dfrac{\partial u}{\partial t}(t,x) - \sup\limits_{v\in U}\{\mathcal{L}(t,x,v)u(t,x) + g(t,x,u(t,x),\nabla u(t,x)\sigma(t,x,v),v)\}\big) = 0, \\
u(T,x) = \Phi(x),
\end{cases} \tag{4.1}
$$

where $\mathcal{L}$ is a family of second order linear partial differential operators,

$$\mathcal{L}(t,x,v)\varphi = \frac{1}{2}Tr\left((\sigma\sigma^T)(t,x,v)D^2\varphi\right) + \langle b(t,x,v), D\varphi\rangle.$$

Here the function b, σ, g, Φ, h are supposed to satisfy (H3.1)–(H3.4), respectively.

We want to prove that the value function $u(t,x)$ introduced by (3.9) is the unique viscosity solution of the obstacle problem for HJB equation (4.1). We first recall the definition of a viscosity solution for HJB equation obstacle problem (4.1) from [4]. Below, S^n will denote the set of $n\times n$ symmetric matrices.

Definition 4.1 *Let $u(t,x)\in C((0,T)\times\mathbb{R}^n)$ and $(t,x)\in(0,T)\times\mathbb{R}^n$. We denote by $\mathcal{P}^{2,+}u(t,x)$ [the "parabolic superjet" of u at (t,x)] the set of triples $(p,q,X)\in\mathbb{R}\times\mathbb{R}^n\times S^n$ which are such that*

$$u(s,y)\le u(t,x)+p(s-t)+\langle q,y-x\rangle+\frac{1}{2}\langle X(y-x),y-x\rangle+o(|s-t|+|y-x|^2).$$

Similarly, we denote by $\mathcal{P}^{2,-}u(t,x)$ [the "parabolic subjet" of u at (t,x)] the set of triples $(p,q,X)\in\mathbb{R}\times\mathbb{R}^n\times S^n$ which are such that

$$u(s,y)\ge u(t,x)+p(s-t)+\langle q,y-x\rangle+\frac{1}{2}\langle X(y-x),y-x\rangle+o(|s-t|+|y-x|^2).$$

Example 4.2 *Suppose that $\varphi\in C^{1,2}((0,T)\times\mathbb{R}^n)$. If $u-\varphi$ has a local maximum at (t,x), then*

$$\left(\frac{\partial\varphi}{\partial t}(t,x),\nabla\varphi(t,x),D^2\varphi(t,x)\right)\in\mathcal{P}^{2,+}u(t,x).$$

If $u-\varphi$ has a local minimum at (t,x), then

$$\left(\frac{\partial\varphi}{\partial t}(t,x),\nabla\varphi(t,x),D^2\varphi(t,x)\right)\in\mathcal{P}^{2,-}u(t,x).$$

We can now give the definition of a viscosity solution of the HJB equation obstacle problem (4.1) .

Definition 4.3

(a) *It can be said $u(t,x)\in C([0,T]\times\mathbb{R}^n)$ is a viscosity subsolution of (4.1) if $u(T,x)\le\Phi(x)$, $x\in\mathbb{R}^n$, and at any point $(t,x)\in(0,T)\times\mathbb{R}^n$, for any $(p,q,X)\in\mathcal{P}^{2,+}u(t,x)$,*

$$\min\left(u(t,x)-h(t,x),-p-\sup_{v\in U}\left\{\frac{1}{2}Tr(aX)+\langle b,q\rangle+g(t,x,u(t,x),q\sigma(t,x,v),v)\right\}\right)\le 0.$$

In other words at any point (t,x) where $u(t,x)>h(t,x)$,

$$-p-\sup_{v\in U}\left\{\frac{1}{2}Tr(aX)+\langle b,q\rangle+g(t,x,u(t,x),q\sigma(t,x,v),v)\right\}\le 0.$$

(b) *It can be said* $u(t,x) \in C([0,T] \times \mathbb{R}^n)$ *is a viscosity supersolution of (4.1) if* $u(T,x) \geq \Phi(x)$, $x \in \mathbb{R}^n$, *and at any point* $(t,x) \in (0,T) \times \mathbb{R}^n$, *for any* $(p,q,X) \in \mathcal{P}^{2,-}u(t,x)$,

$$\min\left(u(t,x) - h(t,x), -p - \sup_{v \in U}\left\{\frac{1}{2}Tr(aX) + \langle b, q\rangle + g(t,x,u(t,x), q\sigma(t,x,v), v)\right\}\right) \geq 0.$$

In other words, at each point, we have both $u(t,x) \geq h(t,x)$ *and*

$$-p - \sup_{v \in U}\left\{\frac{1}{2}Tr(aX) + \langle b, q\rangle + g(t,x,u(t,x), q\sigma(t,x,v), v)\right\} \geq 0.$$

(c) $u(t,x) \in C([0,T] \times \mathbb{R}^n)$ *is said to be a viscosity solution of (4.1) if it is both a viscosity sub- and supersolution.*

We are going to use the approximation of the reflected BSDE by penalization, which was studied in section 6 of [9]. For each $(t,x) \in [0,T] \times \mathbb{R}^n$, $n \in \mathbb{N}$, let $\{({}^nY_s^{t,x;v}, {}^nZ_s^{t,x;v}), t \leq s \leq T\}$ denote the solution of the BSDE

$$\begin{aligned} {}^nY_s^{t,x;v} &= \Phi(X_T^{t,x;v}) + \int_s^T g(r, X_r^{t,x;v}, {}^nY_r^{t,x;v}, {}^nZ_r^{t,x;v}, v_r)dr \\ &\quad + n\int_s^T ({}^nY_r^{t,x;v} - h(r, X_r^{t,x;v}))^- dr - \int_s^T {}^nZ_r^{t,x;v} dW_r, \quad t \leq s \leq T. \end{aligned}$$

We define

$$J_n(t,x;v(\cdot)) := {}^nY_t^{t,x;v}, \qquad v(\cdot) \in \mathcal{U},\ 0 \leq t \leq T,\ x \in \mathbb{R}^n; \tag{4.2}$$

$$u_n(t,x) := ess \sup_{v(\cdot) \in \mathcal{U}} J_n(t,x;v(\cdot)), \qquad 0 \leq t \leq T,\ x \in \mathbb{R}^n. \tag{4.3}$$

It is known from [12] or [13] that $u_n(t,x)$ defined in (4.3) is the viscosity solution of the PDE

$$\begin{cases} -\dfrac{\partial u_n}{\partial t}(t,x) - \sup\limits_{v \in U}\{\mathcal{L}(t,x,v)u_n(t,x) + g_n(t,x,u_n(t,x), \nabla u_n(t,x)\sigma(t,x,v), v)\} = 0, \\ u_n(T,x) = \Phi(x), \end{cases}$$

where

$$g_n(t,x,r,p\sigma(t,x,v),v) = g(t,x,r,p\sigma(t,x,v),v) + n(r - h(t,x))^-.$$

Then

Lemma 4.4 $u_n(t,x) \uparrow u(t,x)$, $0 \leq t \leq T$, $x \in \mathbb{R}^n$.

Proof: From the result of the section 6 in [9], for each $0 \leq t \leq T$, $x \in \mathbb{R}^n$,

$$J_n(t,x;v(\cdot)) \uparrow J(t,x;v(\cdot)), \qquad \text{as} \quad n \to \infty.$$

From the monotonic property of J_n and the definition of u_n in (4.3), we get the monotonic property of u_n. Next we will show the convergent property of u_n.

For each $0 \leq t \leq T$, $x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, there exists $v(\cdot) \in \mathcal{U}$ such that

$$u(t,x) < Y_t^{t,x;v} + \varepsilon,$$

then

$$0 \le u(t,x) - u_n(t,x) \le Y_t^{t,x;v} -^n Y_t^{t,x;v} + \varepsilon.$$

Because ${}^nY_t^{t,x;v} \uparrow Y_t^{t,x;v}$, *a.s.*, we take limit on both side,

$$0 \le \limsup_{n\to\infty}(u(t,x) - u_n(t,x)) \le \varepsilon.$$

From the arbitrariness of ε, we get the desired result. □

Remark 4.5 *Since u_n and u are continuous, it follows from Dini's theorem that the convergence in the lemma is uniform on compacts.*

Theorem 4.6 *Defined by (3.9), u is a viscosity solution of HJB equations (4.1).*

Proof: We now show that u is a subsolution of (4.1). Let (t,x) be a point at which $u(t,x) > h(t,x)$, and let $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$.

From Lemma 6.1 in [4], there exists sequences

$$n_j \to +\infty, \quad (t_j,x_j) \to (t,x), \quad (p_j,q_j,X_j) \in \mathcal{P}^{2,+}u_{n_j}(t_j,x_j),$$

such that

$$(p_j,q_j,X_j) \to (p,q,X).$$

But for any j,

$$\begin{aligned} -p_j - \sup_{v\in U}\Big\{&\frac{1}{2}Tr(aX_j) + \langle b,q_j\rangle + g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v) \\ &+n_j(u_{n_j}(t_j,x_j) - h(t_j,x_j))^-\Big\} \le 0. \end{aligned}$$

From the assumption that $u(t,x) > h(t,x)$ and the uniform convergence of u_n, it follows that for j large enough $u_{n_j}(t_j,x_j) > h(t_j,x_j)$, hence

$$-p_j - \sup_{v\in U}\left\{\frac{1}{2}Tr(aX_j) + \langle b,q_j\rangle + g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v)\right\} \le 0.$$

Let us admit for a moment the following lemma.

Lemma 4.7

$$\begin{aligned} &\lim_{j\to\infty}\sup_{v\in U}\left\{\frac{1}{2}Tr(aX_j) + \langle b,q_j\rangle + g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v)\right\} \\ &= \sup_{v\in U}\lim_{j\to\infty}\left\{\frac{1}{2}Tr(aX_j) + \langle b,q_j\rangle + g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v)\right\}. \end{aligned}$$

Taking the limit as $j \to \infty$ in the above inequality yields:

$$-p - \sup_{v\in U}\left\{\frac{1}{2}Tr(aX) + \langle b,q\rangle + g(t,x,u(t,x),q\sigma(t,x,v),v)\right\} \le 0,$$

and we have proved that u is a subsolution of (4.1).

We now show that u is a supersolution of (4.1). Let (t,x) be an arbitrary point in $(0,T)\times\mathbb{R}^n$, and $(p,q,X)\in\mathcal{P}^{2,-}u(t,x)$. We already know that $u(t,x)\geq h(t,x)$. By the same argument as above, there exist sequences:

$$n_j\to+\infty,\quad (t_j,x_j)\to(t,x),\quad (p_j,q_j,X_j)\in\mathcal{P}^{2,-}u_{n_j}(t_j,x_j),$$

such that

$$(p_j,q_j,X_j)\to(p,q,X).$$

But for any j,

$$\begin{aligned}&-p_j-\sup_{v\in U}\Big\{\frac{1}{2}Tr(aX_j)+\langle b,q_j\rangle+g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v)\\&\qquad\qquad+n_j(u_{n_j}(t_j,x_j)-h(t_j,x_j))^-\Big\}\geq 0.\end{aligned}$$

Hence

$$-p_j-\sup_{v\in U}\left\{\frac{1}{2}Tr(aX_j)+\langle b,q_j\rangle+g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v)\right\}\geq 0,$$

and taking the limit as $j\to\infty$, we conclude that:

$$-p-\sup_{v\in U}\left\{\frac{1}{2}Tr(aX)+\langle b,q\rangle+g(t,x,u(t,x),q\sigma(t,,v),v)\right\}\geq 0.$$

□

Now we turn to

Proof of Lemma 4.7 For the convenience, we denote

$$f_j(v)=\frac{1}{2}Tr(a,X_j)+\langle b,q_j\rangle+g(t_j,x_j,u_{n_j}(t_j,x_j),q_j\sigma(t_j,x_j,v),v).$$

Firstly, $\forall v\in U$,

$$f_j(v)\leq\sup_{v\in U}f_j(v),\qquad \lim_{j\to\infty}f_j(v)\leq\liminf_{j\to\infty}\sup_{v\in U}f_j(v),$$

then

$$\sup_{v\in U}\lim_{j\to\infty}f_j(v)\leq\liminf_{j\to\infty}\sup_{v\in U}f_j(v).\tag{4.4}$$

Secondly, we consider a subsequence $\{j_k\}_{k=1}^{\infty}$ such that

$$\lim_{j_k\to\infty}\sup_{v\in U}f_{j_k}(v)=\limsup_{j\to\infty}\sup_{v\in U}f_j(v).$$

$\forall\varepsilon>0$, $\forall j_k$, $\exists v_{j_k}\in U$ such that

$$\sup_{v\in U}f_{j_k}(v)\leq f_{j_k}(v_{j_k})+\varepsilon.$$

Because U is compact, there exists a convergent subsequence denoted by $\{v_{j_k}\}_{k=1}^{\infty}$ also, the limit is denoted by v_0. We consider the difference of $f_{j_k}(v_{j_k})$ and $f_{j_k}(v_0)$: From the Lipschitz condition we get

$$|f_{j_k}(v_{j_k}) - f_{j_k}(v_0)| \leq C|v_{j_k} - v_0|^2 + C|v_{j_k} - v_0|,$$

where C only depend on the Lipschitz constant. It follows that for j_k large enough

$$|f_{j_k}(v_{j_k}) - f_{j_k}(v_0)| \leq \varepsilon.$$

Then

$$\sup_{v\in U} f_{j_k}(v) \leq f_{j_k}(v_0) + 2\varepsilon,$$

$$\limsup_{j\to\infty} \sup_{v\in U} f_j(v) = \lim_{j_k\to\infty} \sup_{v\in U} f_{j_k}(v) \leq \lim_{j_k\to\infty} f_{j_k}(v_0) + 2\varepsilon = \lim_{j\to\infty} f_j(v_0) + 2\varepsilon,$$

$$\limsup_{j\to\infty} \sup_{v\in U} f_j(v) \leq \sup_{v\in U} \lim_{j\to\infty} f_j(v_0) + 2\varepsilon.$$

From the arbitrariness of ε,

$$\limsup_{j\to\infty} \sup_{v\in U} f_j(v) \leq \sup_{v\in U} \lim_{j\to\infty} f_j(v_0). \tag{4.5}$$

From (4.4) and (4.5), we complete the proof. □

Finally, we shall use some technique and method from [1] to establish a uniqueness result for viscosity solution of (4.1). This kind of technique and method can also be seen in [3] to prove the uniqueness for viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations related to stochastic differential games.

Lemma 4.8 *Let $u_1 \in C([0,T]\times\mathbb{R}^n)$ be a viscosity subsolution and $u_2 \in C([0,T]\times\mathbb{R}^n)$ be a viscosity supersolution of (4.1). Then the function $w := u_1 - u_2$ is a viscosity subsolution of the system*

$$\begin{cases} \min\left(w(t,x), -\dfrac{\partial w}{\partial t}(t,x) - \sup\limits_{v\in U}\{\mathcal{L}(t,x,v)w(t,x) + L|w| + L|\nabla w\sigma(t,x,v)|\}\right) = 0, \\ w(T,x) = 0, \end{cases} \tag{4.6}$$

where L is the Lipschitz constant of g in (y,z).

Proof: The proof is similar to that of the corresponding results: Lemma 3.7 in [1].

For each $(t_0,x_0) \in (0,T)\times\mathbb{R}^n$, let $\varphi \in C^\infty([0,T]\times\mathbb{R}^n)$ and let (t_0,x_0) be a strict global maximum point of $w-\varphi$. Because u_2 is a viscosity supersolution of HJB equation (4.1), we have $u_2(t_0,x_0) \geq h(t_0,x_0)$. If $u_1(t_0,x_0) \leq h(t_0,x_0)$, it is easily to get

$$w(t_0,x_0) = u_1(t_0,x_0) - u_2(t_0,x_0) \leq 0,$$

and we get the desired result. Therefore, in the proof, we suppose that $u(t_0,x_0) > h(t_0,x_0)$.

We introduce the function

$$\Phi_\varepsilon(t,x,y) = u_1(t,x) - u_2(t,y) - \frac{|x-y|^2}{\varepsilon^2} - \varphi(t,x),$$

where ε is a positive parameter which is devoted to tend to zero.

Since (t_0, x_0) is a strict global maximum point of $u_1 - u_2 - \varphi$, by a classical argument in the theory of viscosity solutions, there exists a sequence $(\hat{t}, \hat{x}, \hat{y})$ such that

(i) $(\hat{t}, \hat{x}, \hat{y})$ is a global maximum point of Φ_ε in $[0,T] \times \bar{B}_R \times \bar{B}_R$ where B_R is a ball with a large radius R;

(ii) $(\hat{t}, \hat{x})$, $(\hat{t}, \hat{y}) \to (t_0, x_0)$ as $\varepsilon \to 0^+$;

(iii) $\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2}$ is bounded and tend to zero when $\varepsilon \to 0^+$.

We have dropped above the dependence of $\hat{t}$, $\hat{x}$ and $\hat{y}$ in ε for the sake of simplicity of notations.

It follows from Theorem 8.3 in [4] that, $\forall \delta > 0$, there exist

$$\left(p, \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + D\varphi, X\right) \in \bar{\mathcal{P}}^{2,+} u_1(\hat{t}, \hat{x}), \quad \left(p - \frac{\partial \varphi}{\partial t}, \frac{2(\hat{x}-\hat{y})}{\varepsilon^2}, Y\right) \in \bar{\mathcal{P}}^{2,-} u_2(\hat{t}, \hat{y}),$$

such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \delta A^2, \tag{4.7}$$

where

$$A = \begin{pmatrix} \frac{2}{\varepsilon^2} + D^2\varphi & -\frac{2}{\varepsilon^2} \\ -\frac{2}{\varepsilon^2} & \frac{2}{\varepsilon^2} \end{pmatrix}.$$

Calculating directly, we get

$$\begin{aligned} A + \delta A^2 = & \left(\frac{2}{\varepsilon^2} + \delta \frac{4}{\varepsilon^4}\right) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \left(1 + \delta \frac{4}{\varepsilon^2}\right) \begin{pmatrix} D^2\varphi & 0 \\ 0 & 0 \end{pmatrix} \\ & + \delta \frac{4}{\varepsilon^4} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \delta \begin{pmatrix} (D^2\varphi)^2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

After given ε, $\delta > 0$, we have

$$\begin{aligned} & -p - \sup_{v \in U} \left\{ \frac{1}{2} Tr\left((\sigma\sigma^T)(\hat{t}, \hat{x}, v) X\right) + \langle b(\hat{t}, \hat{x}, v), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + D\varphi(\hat{t}, \hat{x}) \rangle \right. \\ & \qquad \left. + g\left(\hat{t}, \hat{x}, u_1(\hat{t}, \hat{x}), [\frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + D\varphi(\hat{t}, \hat{x})]\sigma(\hat{t}, \hat{x}, v), v\right) \right\} \leq 0, \\ & -\left(p - \frac{\partial \varphi}{\partial t}(\hat{t}, \hat{x})\right) - \sup_{v \in U} \left\{ \frac{1}{2} Tr\left((\sigma\sigma^T)(\hat{t}, \hat{y}, v) Y\right) + \langle b(\hat{t}, \hat{y}, v), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} \rangle \right. \\ & \qquad \left. + g\left(\hat{t}, \hat{y}, u_2(\hat{t}, \hat{y}), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} \sigma(\hat{t}, \hat{y}, v), v\right) \right\} \geq 0. \end{aligned}$$

The first inequality minus the second one,

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(\hat{t}, \hat{x}) - \sup_{v \in U} \left\{ \frac{1}{2} \left(Tr\left((\sigma\sigma^T)(\hat{t}, \hat{x}, v) X\right) - Tr\left((\sigma\sigma^T)(\hat{t}, \hat{y}, v) Y\right) \right) \right. \\ & + \left(\langle b(\hat{t}, \hat{x}, v), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + D\varphi(\hat{t}, \hat{x}) \rangle - \langle b(\hat{t}, \hat{y}, v), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} \rangle \right) \\ & \left. + \left[g\left(\hat{t}, \hat{x}, u_1(\hat{t}, \hat{x}), [\frac{2(\hat{x}-\hat{y})}{\varepsilon^2} + D\varphi(\hat{t}, \hat{x})]\sigma(\hat{t}, \hat{x}, v)\right) - g\left(\hat{t}, \hat{y}, u_2(\hat{t}, \hat{y}), \frac{2(\hat{x}-\hat{y})}{\varepsilon^2} \sigma(\hat{t}, \hat{y}, v)\right) \right] \right\} \\ & \leq 0. \end{aligned}$$

Using (4.7) and Lipschitz condition, we analysis the items in the $\sup_{v\in U}$ and get

$$
\begin{aligned}
&-\frac{\partial\varphi}{\partial t}(\hat{t},\hat{x})-\sup_{v\in U}\Big\{\frac{1}{2}(\frac{2}{\varepsilon^2}+\delta\frac{4}{\varepsilon^4})L^2|\hat{x}-\hat{y}|^2+\frac{1}{2}(1+\delta\frac{4}{\varepsilon^2})Tr\left((\sigma\sigma^T)(\hat{t},\hat{x},v)D^2\varphi(\hat{t},\hat{x})\right)\\
&+\frac{1}{2}\delta\frac{4}{\varepsilon^4}(|\sigma(\hat{t},\hat{x},v)|^2+|\sigma(\hat{t},\hat{y},v)|^2)+\frac{1}{2}\delta Tr\left((\sigma\sigma^T)(\hat{t},\hat{x},v)(D^2\varphi)^2(\hat{t},\hat{x})\right)\\
&+2L\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2}+\langle b(\hat{t},\hat{x},v),D\varphi(\hat{t},\hat{x})\rangle+L|\hat{x}-\hat{y}|+L|u_1(\hat{t},\hat{x})-u_2(\hat{t},\hat{x})|\\
&+L|u_2(\hat{t},\hat{x})-u_2(\hat{t},\hat{y})|+L|D\varphi(\hat{t},\hat{x})\sigma(\hat{t},\hat{x},v)|+2L^2\frac{|\hat{x}-\hat{y}|^2}{\varepsilon^2}\Big\}\le 0.
\end{aligned}
$$

We let $\delta\to 0^+$, then let $\varepsilon\to 0^+$ and we get

$$
\begin{aligned}
-\frac{\partial\varphi}{\partial t}(t_0,x_0)-\sup_{v\in U}\Big\{&\frac{1}{2}Tr\left((\sigma\sigma^T)(t_0,x_0,v)D^2\varphi(t_0,x_0)\right)+\langle b(t_0,x_0,v),D\varphi(t_0,x_0)\rangle\\
&+L|w(t_0,x_0)|+L|D\varphi(t_0,x_0)\sigma(t_0,x_0,v)|\}\le 0.
\end{aligned}
$$

Therefore w is a viscosity subsolution of the desired equation (4.6) and the proof is completed. $\square$

Now we are going to construct one suitable smooth supersolution for the equation (4.6).

Lemma 4.9 *For any $A>0$, there exists $C_1>0$ such that the function*

$$\chi(t,x)=\exp\{(C_1(T-t)+A)\psi(x)\},$$

where

$$\psi(x)=\left[\log\left((|x|^2+1)^{\frac{1}{2}}\right)+1\right]^2$$

satisfies

$$\min\left(\chi(t,x),-\frac{\partial\chi}{\partial t}(t,x)-\sup_{v\in U}\{\mathcal{L}(t,x,v)\chi(t,x)+L\chi(t,x)+L|\nabla\chi\sigma(t,x,v)|\}\right)>0$$

in $[t_1,T]\times\mathbb{R}^n$ where $t_1=T-(A/C_1)$.

Proof: Obviously, the function χ defined in the Lemma satisfy $\chi(t,x)>0$, for each $(t,x)\in[0,T]\times\mathbb{R}^n$. We give estimations on the first and second order derivatives of ψ:

$$|D\psi(x)|\le\frac{2[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}}\quad\textit{and}\quad|D^2\psi(x)|\le\frac{C\left(1+[\psi(x)]^{\frac{1}{2}}\right)}{|x|^2+1}\quad\textit{in}\quad\mathbb{R}^n.$$

These estimations imply that, if $t\in[t_1,T]$,

$$|D\chi(t,x)|\le C\chi(t,x)\frac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}},\quad|D^2\chi(t,x)|\le C\chi(t,x)\frac{\psi(x)}{|x|^2+1},$$

where the constant C only depend on A. We continue to calculate

$$\begin{aligned}
&\frac{\partial \chi}{\partial t}(t,x)+\sup_{v\in U}\{\mathcal{L}(t,x,v)\chi(t,x)+L\chi(t,x)+L|\nabla\chi\sigma(t,x,v)|\}\\
&=\frac{\partial \chi}{\partial t}(t,x)+\sup_{v\in U}\left\{\frac{1}{2}Tr((\sigma\sigma^T)D^2\chi)+\langle b,D\chi\rangle+L\chi(t,x)+L|\nabla\chi\sigma(t,x,v)|\right\}\\
&\leq -C_1\chi(t,x)\psi(x)+\sup_{v\in U}\left\{\frac{1}{2}\frac{|\sigma(t,x,v)|^2}{|x|^2+1}C\chi(t,x)\psi(x)\right.\\
&\quad\left.+\frac{|b(t,x,v)|}{(|x|^2+1)^{\frac{1}{2}}}C\chi(t,x)[\psi(x)]^{\frac{1}{2}}+L\chi(t,x)+L\frac{|\sigma(t,x,v)|}{(|x|^2+1)^{\frac{1}{2}}}C\chi(t,x)[\psi(x)]^{\frac{1}{2}}\right\}.
\end{aligned}\tag{4.8}$$

Because b and σ are linear growth in x, $[\psi(x)]^{\frac{1}{2}}\leq\psi(x)$ and $1\leq\psi(x)$, the above inequality (4.8)

$$\begin{aligned}
&< -C_1\chi(t,x)\psi(x)+\frac{1}{2}C\chi(t,x)\psi(x)+C\chi(t,x)\psi(x)+L\chi(t,x)\psi(x)+LC\chi(t,x)\psi(x)\\
&=-(C_1-\frac{1}{2}C-C-L-LC)\chi(t,x)\psi(x).
\end{aligned}$$

It is clear that when C_1 large enough the quantity in the right side of the above inequality is negative and the proof is completed. $\square$

Now we can prove the uniqueness result for viscosity solution of (4.1).

Theorem 4.10 *Assume that b, σ, g, Φ and h satisfy (H3.1)–(H3.4), respectively. Then there exists at most one viscosity solution of HJB equation (4.1) in the class of continuous functions which grow at most polynomially at infinity.*

Proof: Let $u_1,u_2\in C([0,T]\times\mathbb{R}^n)$ be two viscosity solutions of HJB equation (4.1).
We define $w:=u_1-u_2$, then we have

$$\lim_{|x|\to\infty}w(t,x)e^{-A[\log((|x|^2+1)^{\frac{1}{2}})]^2}=0$$

uniformly for $t\in[0,T]$, for some $A>0$. This implies, in particular, that $w(t,x)-\alpha\chi(t,x)$ is bounded from above in $[t_1,T]\times\mathbb{R}^n$ for any $\alpha>0$ and that

$$M:=\max_{[t_1,T]\times\mathbb{R}^n}(w-\alpha\chi)(t,x)e^{-L(T-t)}$$

is achieved at some point $(t_0,x_0)\in[t_1,T]\times\mathbb{R}^n$ (depend on α). Then we have two case.

The first case: $w(t_0,x_0)\leq 0$.
Then we have

$$u_1(t,x)-u_2(t,x)\leq\alpha\chi(t,x),\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n.$$

Letting α tends to zero, we obtain

$$u_1(t,x)\leq u_2(t,x),\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n.\tag{4.9}$$

The second case: $w(t_0,x_0)>0$.
Then we have

$$w(t,x)-\alpha\chi(t,x)\le (w(t_0,x_0)-\alpha\chi(t_0,x_0))e^{-L(t-t_0)},\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n.$$

We define

$$\varphi(t,x)=\alpha\chi(t,x)+(w(t_0,x_0)-\alpha\chi(t_0,x_0))e^{-L(t-t_0)},$$

and can get

$$w-\varphi\le 0=(w-\varphi)(t_0,x_0),\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n.$$

Since $\varphi(t_0,x_0)=w(t_0,x_0)>0$ and Lemma 4.8, when $t_0\in[t_1,T)$, we have

$$\begin{aligned}&-\frac{\partial\varphi}{\partial t}(t_0,x_0)-\sup_{v\in U}\Big\{\frac{1}{2}Tr\left((\sigma\sigma^T)(t_0,x_0,v)D^2\varphi(t_0,x_0)\right)+\langle b(t_0,x_0,v),D\varphi(t_0,x_0)\rangle\\&\quad+L\varphi(t_0,x_0)+L|\nabla\varphi(t_0,x_0)\sigma(t_0,x_0,v)|\Big\}\le 0.\end{aligned}$$

From the definition of φ, we rewrite the above inequality

$$\begin{aligned}&\alpha\Big[-\frac{\partial\chi}{\partial t}(t_0,x_0)-\sup_{v\in U}\Big\{\frac{1}{2}Tr\left((\sigma\sigma^T)(t_0,x_0,v)D^2\chi(t_0,x_0)\right)+\langle b(t_0,x_0,v),D\chi(t_0,x_0)\rangle\\&\quad+L\chi(t_0,x_0)+L|\nabla\chi(t_0,x_0)\sigma(t_0,x_0,v)|\Big\}\Big]\le 0.\end{aligned}$$

This is a contradiction with Lemma 4.9. Therefore $t_0=T$, this is a contradiction with the fact that $w(t,x)$ is a viscosity subsolution of (4.6) (see Lemma 4.8). Then the second case does not happen.

If we change $w(t,x)=u_1-u_2$ for $w'(t,x)=u_2-u_1$, the same argument leads to

$$u_2(t,x)\le u_1(t,x),\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n. \tag{4.10}$$

Combining (4.9) with (4.10), we have

$$u_1(t,x)=u_2(t,x),\qquad (t,x)\in[t_1,T]\times\mathbb{R}^n.$$

Applying successively the same argument on the intervals $[t_2,t_1]$ where $t_2=(t_1-A/C_1)^+$ and then, if $t_2>0$ on $[t_3,t_2]$ where $t_3=(t_2-A/C_1)^+$ ... etc. We finally obtain that

$$u_1(t,x)=u_2(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

The proof is complete. □

Appendix

In the appendix we give the proof of Proposition 2.1 and 2.2.

Proof of Proposition 2.1

Applying Itô's formula to the process $|Y_s|^2e^{\beta s}$ yields

$$\begin{aligned}
&|Y_t|^2e^{\beta t}+\int_t^T(\beta|Y_s|^2+|Z_s|^2)e^{\beta s}ds\\
&=|\xi|^2e^{\beta T}+2\int_t^T Y_sg(s,Y_s,Z_s)e^{\beta s}ds+2\int_t^T Y_se^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s\\
&=|\xi|^2e^{\beta T}+2\int_t^T Y_sg(s,Y_s,Z_s)e^{\beta s}ds+2\int_t^T S_se^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s,
\end{aligned}$$

where we have used the identity $\int_t^T(Y_s-S_s)e^{\beta s}dK_s=0$. Using the Lipschitz property of g, we have

$$\begin{aligned}
&|Y_t|^2e^{\beta t}+\int_t^T(\beta|Y_s|^2+|Z_s|^2)e^{\beta s}ds\\
&\le|\xi|^2e^{\beta T}+2\int_t^T |Y_s||g(s,Y_s,Z_s)|e^{\beta s}ds+2\int_t^T |S_s|e^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s\\
&\le|\xi|^2e^{\beta T}+2\int_t^T |Y_s||g(s,0,0)|e^{\beta s}ds+2\int_t^T (L|Y_s|^2+L|Y_s||Z_s|)e^{\beta s}ds\\
&\quad+2\int_t^T |S_s|e^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s\\
&\le|\xi|^2e^{\beta T}+2\int_t^T |Y_s||g(s,0,0)|e^{\beta s}ds+\int_t^T\left((2L+2L^2)|Y_s|^2+\frac{1}{2}|Z_s|^2\right)e^{\beta s}ds\\
&\quad+2\int_t^T |S_s|e^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s.
\end{aligned}$$

We select $\beta=2L^2+2L$, then

$$\begin{aligned}
|Y_t|^2e^{\beta t}+\frac{1}{2}\int_t^T|Z_s|^2e^{\beta s}ds\le|\xi|^2e^{\beta T}+2\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds\\
+2\int_t^T|S_s|e^{\beta s}dK_s-2\int_t^T Y_sZ_se^{\beta s}dW_s.
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t}\left\{\int_t^T|Z_s|^2e^{\beta s}ds\right\}\le 2\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2e^{\beta T}+2\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds\right.\\
\left.+2\int_t^T|S_s|e^{\beta s}dK_s\right\}.
\end{aligned}\tag{A.2}$$

$$\begin{aligned}
\sup_{t\le u\le T}|Y_u|^2e^{\beta u}\le|\xi|^2e^{\beta T}+2\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds\\
+2\int_t^T|S_s|e^{\beta s}dK_s+4\sup_{t\le u\le T}|\int_t^u Y_sZ_se^{\beta s}dW_s|.
\end{aligned}$$

From Burkholder-Davis-Gundy's inequality we have

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}\right\}\le \mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2e^{\beta T}+2\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds\right.$$
$$\left.+2\int_t^T|S_s|e^{\beta s}dK_s\right\}+C\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T|Y_s|^2|Z_s|^2e^{2\beta s}ds\right)^{\frac{1}{2}},$$

thanks to the inequality $ab\le a^2/2+b^2/2$, we deduce immediately

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}\right\}\le \mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2e^{\beta T}+2\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds+2\int_t^T|S_s|e^{\beta s}dK_s\right\}$$
$$+\frac{C^2}{2}\mathbb{E}^{\mathcal{F}_t}\left\{\int_t^T|Z_s|^2e^{\beta s}ds\right\}+\frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}\right\}.$$

Combining the inequality (A.2) with the above one, we easily derive that

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}+\int_t^T|Z_s|^2e^{\beta s}ds\right\}$$
$$\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2e^{\beta T}+\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds+2\int_t^T|S_s|e^{\beta s}dK_s\right\}.$$

Using the fact that

$$C\mathbb{E}^{\mathcal{F}_t}\left\{\int_t^T|Y_s||g(s,0,0)|e^{\beta s}ds\right\}\le\frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}\right\}+\frac{C^2}{2}\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T|g(s,0,0)|e^{(\beta/2)s}ds\right)^2,$$

we get

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2e^{\beta u}+\int_t^T|Z_s|^2e^{\beta s}ds\right\}$$
$$\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2e^{\beta T}+\left(\int_t^T|g(s,0,0)|e^{(\beta/2)s}ds\right)^2+2\int_t^T|S_s|e^{\beta s}dK_s\right\}.$$

Then we drop the exponential function to get a brief form

$$\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|Y_u|^2+\int_t^T|Z_s|^2ds\right\}$$
$$\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2+\left(\int_t^T|g(s,0,0)|ds\right)^2+2\int_t^T|S_s|dK_s\right\}. \tag{A.3}$$

We now give an estimate of $\mathbb{E}^{\mathcal{F}_t}[|K_T - K_t|^2]$. From the equation

$$K_T - K_t = Y_t - \xi - \int_t^T g(s, Y_s, Z_s)ds + \int_t^T Z_s dW_s,$$

and estimate (A.3), we get the following inequalities:

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_t}\left\{|K_T - K_t|^2\right\} &\leq C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2 + \left(\int_t^T |g(s,0,0)|ds\right)^2 + 2\int_t^T |S_s|dK_s\right\} \\
&\leq C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2 + \left(\int_t^T |g(s,0,0)|ds\right)^2\right\} \\
&\quad + 2C^2\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\leq u\leq T}|S_s|^2\right\} + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\left\{|K_T - K_t|^2\right\}.
\end{aligned}$$

Consequently,

$$\mathbb{E}^{\mathcal{F}_t}\left\{|K_T - K_t|^2\right\} \leq C\mathbb{E}^{\mathcal{F}_t}\left\{|\xi|^2 + \left(\int_t^T |g(s,0,0)|ds\right)^2 + \sup_{t\leq u\leq T}|S_s|^2\right\}. \tag{A.4}$$

Combining the estimate (A.3) with (A.4), we complete the proof of the proposition. $\square$

Proof of Proposition 2.2

The computation process is similar to that in the proof of Proposition 2.1, so we shall only give the sketch of the proof. Since $\int_t^T (\Delta Y_s - \Delta S_s)e^{\beta s}d(\Delta K_s) \leq 0$,

$$\begin{aligned}
&|\Delta Y_t|e^{\beta t} + \int_t^T (\beta|\Delta Y_s|^2 + |\Delta Z_s|^2)e^{\beta s}ds \\
&\leq |\Delta\xi|^2 e^{\beta T} + 2\int_t^T \Delta Y_s \Delta g(s, Y_s, Z_s)e^{\beta s}ds \\
&\quad + 2\int_t^T \Delta Y_s[g'(s, Y_s, Z_s) - g'(s, Y_s', Z_s')]e^{\beta s}ds \\
&\quad + 2\int_t^T \Delta S_s e^{\beta s}d(\Delta K_s) - 2\int_t^T \Delta Y_s \Delta Z_s e^{\beta s}dW_s \\
&\leq |\Delta\xi|^2 e^{\beta T} + 2\int_t^T |\Delta Y_s||\Delta g(s, Y_s, Z_s)|e^{\beta s}ds \\
&\quad + 2L\int_t^T (|\Delta Y_s|^2 + |\Delta Y_s||\Delta Z_s|)e^{\beta s}ds \\
&\quad + 2\int_t^T |\Delta S_s|e^{\beta s}d(K_s + K_s') - 2\int_t^T \Delta Y_s \Delta Z_s e^{\beta s}dW_s.
\end{aligned}$$

Similar technique with the above proof of Proposition 2.1, we can get

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|\Delta Y_u|^2+\int_t^T|\Delta Z_s|^2ds\right\}\\
&\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\Delta\xi|^2+\left(\int_t^T|\Delta g(s,Y_s,Z_s)|ds\right)^2+2\int_t^T|\Delta S_s|d(K_s+K_s')\right\}\\
&\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\Delta\xi|^2+\left(\int_t^T|\Delta g(s,Y_s,Z_s)|ds\right)^2+\left(\sup_{t\le u\le T}|\Delta S_u|\right)((K_T-K_t)+(K_T'-K_t'))\right\}\\
&\le C\mathbb{E}^{\mathcal{F}_t}\left\{|\Delta\xi|^2+\left(\int_t^T|\Delta g(s,Y_s,Z_s)|ds\right)^2\right\}\\
&\quad+\left(\mathbb{E}^{\mathcal{F}_t}\left\{\sup_{t\le u\le T}|\Delta S_u|^2\right\}\right)^{1/2}\left(\mathbb{E}^{\mathcal{F}_t}\left\{((K_T-K_t)-(K_T'-K_t'))^2\right\}\right)^{1/2}.
\end{aligned}
$$

And then from Proposition 2.1, we complete the proof. $\square$

Acknowledgements. The authors express their gratitude to Prof. Shige Peng for his elicitation and inspiring idea in recursive stochastic dynamic programming principle. The authors also thank Dr. Juan Li and Mingyu Xu for their helpful discussions and suggestions.

Reference

[1] G. Barles, R. Buckdahn & E. Pardoux, Backward Stochastic Differential Equations and Integral-Partial Differential Equations, Stochastics and Stochastics Reports, 60(1997), pp. 57-83.

[2] P. Briand, F. Coquet, Y. Hu, J. Mémin & S. Peng, A converse comparison theorem for BSDEs and related properties of $g-$expectation, Elect. Comm. in Probab., 5(2000), pp. 101-117.

[3] R. Buckdahn & J. Li, Stochastic Differential games and Viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs Equations, Preprint, 2006.

[4] M. G. Crandall, H. Ishii & P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Soc., 27(1992), pp. 1-67.

[5] J.Cvitanic & I.Karatzas, Backward SDE's with reflection and Dynkin Games, The Annals of Probability, 24(1996), pp. 2024-2056.

[6] D. Duffie & L. Epstein, Stochastic differential utility, Econometrica, 60(1992), pp. 353–394.

[7] S.Hamadène & J.-P.Lepeltier, Reflected BSDEs and mixed game problems, Stochastic processes and their applications, 85(2000), pp. 177-188.

[8] S.Hamadène, J.-P.Lepeltier & Z.Wu, Infinite horizon Reflected BSDEs and applications in mixed control and game problems, Probability and mathematical statistics, 19(1999), pp. 211-234.

[9] El. Karoui, C. Kapoudjian, E. Pardoux, S. Peng & M.C. Quenez , Reflected solutions of backward SDE's, and related obstacle problems for PDE's, The Annals of Probability, 25(1997), pp. 702-737.

[10] N. El.Karoui, S. Peng & M.C. Quenez, Backward Stochastic Differential Equation in Finance, Math. Finance, 7(1997), pp. 1-71.

[11] E.Pardoux & S.Peng, Adapted solutions of a backward stochastic differential equation, Systems and Control Letters, 14(1990), pp. 55-61.

[12] S.Peng, A generalized dynamic programming principle and Hamilton-Jacobi-Bellmen equation, Stochastics and Stochastic Reports, 38(1992), pp. 119-134.

[13] J. Yan, S.Peng, S.Fang & L.Wu, Topics on stochastic analysis, Science Press. Beijing (in Chinese), 1997.